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A Modal Characterization of λ-Bisimilarity*

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Abstract In order to describe approximate equivalence among processes, a notion of λ-bisimilarity has been introduced in the field of process algebra. In this paper, we provide a modal characterization of λ- bisimilarity without the assumption that the metric is an ultra-metric or $\lambda = 0$, which is a generalization of one obtained by [M. Ying, *Theoret. Comput. Sci.*, Vol. 275, 1–68].

Key words: process algebra; Hennessy-Milner logical characterization; λ-bisimilarity

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1 Introduction

In order to describe approximate equivalence among processes (programs, systems), a number of theories have been presented in different contexts (for example[2,4,9−13]). In these work, the notion of distance plays a central role. Two kinds of distances have been considered in the literature: distances between actions and distances between states.

Based on metric spaces over actions, Ying introduces λ-bisimilarity[11,12] and van Breugel provides three avenues to define behavioural pseudometrics[4]. Here, λ-bisimilarity is a looser bisimilarity, and the behavioural pseudometric is a distance function between states, which is a quantitative analogue of bisimilarity.

Both Ying's and van Breugel's work are based on labelled transition systems (LTS) and metric spaces of actions. Contrastively, the theory presented by de Alfaro et al. does not refer to actions[2]. They introduce the notion of quantitative transition systems (QTS), which refers to a finite set of propositions explicitly and interprets propositions as real values in[0, 1]. In this framework, de Alfaro et al. define the distance between traces and lift it to distance between states. In particular, they introduce the notion of branching distance, which generalizes the notion of bisimilarity. Recently, Pola et al. introduce the notion of metric transition systems, which

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contains an output set endowed with a metric and assigns to each state an output[9,10]. Pola et al. regard metric transition systems as the abstract models of control systems. For metric transition systems with the same output set and metric, the notions of approximate bisimilarity are provided to capture the equivalence between them[9,10].

Each of the models mentioned above has its own strong point. For example, QTSs provide formal models for optimization problems and discrete−time samplings of continuous systems[2]. λ-bisimilarity can be used to analyze approximate implementations of real time systems[11]. Recently, this notion has been extended and adopted to capture the equivalence of processes in π_N[13].

It is well known that bisimilarity can be characterized as a fixed point[8], in terms of a modal logic[6] and by means of a coalgebra[1]. The logical characterization of bisimilarity is obtained by Hennessy and Milner[6], which reveals that two states in a LTS are bisimilar if and only if they satisfy the same formulae of the Hennessy−Milner logic (HML, for short). Moreover, van Benthem et al. explore the connection between modal logic and process theory in a more systematic way[3].

Motivated by the work of Hennessy and Milner, the literature[2,4,11,12] also establish logical characterizations of λ-bisimilarity, behavioural pseudometric and branching distance, respectively.

De Alfaro et al. obtain a logical characterization of branching distances in terms of quantitative μ−calculus[2]. Van Breugel gets a logical characterization of his behavioural pseudometric in terms of formulae of HML$^+$, where HML$^+$ is obtained from HML by adding formulae with the form $\varphi + \lambda$ ($\lambda \in [0, +\infty]$). It should be pointed out that both de Alfaro et al. and van Breugel adopt real−valued interpretations, that is, the value of any formula is a nonnegative real value instead of boolean value.

Given an assumption that the metric is an ultra−metric or $\lambda = 0$, the logical characterization of λ-bisimilarity has been obtained in[11, 12]. But this leaves the open problem of providing logical characterization of λ-bisimilarity in general case [12, p.158, lines 17−18]. In fact, the restriction to ultra−metric is rigorous. Some natural metrics in realistic applications are not ultra−metrics. For instance, as mentioned above, λ-bisimilarity can be adopted to analyze real time systems, however, van Breugel pointed out[4]: since the metrics on the actions of timed transition systems are usually the Euclidean metrics and the Euclidean metrics are not ultra−metrics, logical characterization obtained in[11, 12] may not be very suitable for approximate reasoning of timed transition systems. More such examples may be found in Sections 1, 7 and 8 in[11]. Thus a general logical characterization of λ-bisimilarity is helpful to its application. This paper will provide such a logical characterization of λ-Bisimilarity without the assumption that the metric is an ultra−metric or $\lambda = 0$.

The rest of this paper is organized as follows. In Section 2, we recall related definitions and results in the literature. Section 3 establishes a logical characterization of λ-bisimilarity associated with a general metric. In Section 4, we consider logical characterization of λ-bisimilarity in the ultra−metric case. Section 5 concludes the paper with a short discussion.

2 Preliminaries

2.1 λ-Bisimilarity

The notion of bisimilarity is an important concept in process calculus[7,8]. Roughly speaking, two processes are said to be bisimilar if they can perform the same actions to reach bisimilar states. However, in some circumstances, it is rigorous to require that the actions can be matched only when they are identical. For example, in real time systems, there is often a bit difference between time delays. Thus the notion of bisimilarity may not be very suitable for describing such approximate equivalence. Ying presents the notion of λ-bisimilarity to overcome this defect[11,12]. To define this notion formally, Ying refers to metrics over actions. Recall that a metric space is a pair (X, ρ) in which X is a nonempty set and ρ is a mapping from $X \times X$ into $[0, \infty)$ such that for any $x, y, z \in X$,

(1) $\rho(x, y) = 0$ if and only if $x = y$,

(2) $\rho(x, y) = \rho(y, x)$,

(3) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

If (1) is weakened by

$(1)'$ $\rho(x, x) = 0$ for each $x \in X$,

then ρ is called a pseudometric; and if (3) is strengthened by

$(3)'$ $\rho(x, z) \leqslant \max\{\rho(x, y), \rho(y, z)\}$ for any $x, y, z \in X$,

then ρ is called an ultra−metric.

Definition 1. A LTS $\sigma = (S, A, \{\stackrel{a}{\rightarrow}: a \in A\})$ consists of a set S of states, a set A of labels, and $\stackrel{a}{\rightarrow}$ is a binary relation of S for each $a \in A$.

Definition 2[11]. Let $\sigma = (S, A, \{\stackrel{a}{\rightarrow}: a \in A\})$ be a LTS, ρ a metric on A, $R \subseteq S \times S$ and $\lambda \in [0, \infty)$. The relation R is a λ-bisimulation if and only if for each $(s, t) \in R$, for each $\theta > \lambda$ and for each $a \in A$,

(1) whenever $s \stackrel{a}{\rightarrow} s_1$ then there exist $b \in A$ and $t_1 \in S$ such that $t \stackrel{b}{\rightarrow} t_1$, $\rho(a, b) < \theta$ and $(s_1, t_1) \in R$ (*the forth condition*);

(2) whenever $t \stackrel{a}{\rightarrow} t_1$ then there exist $b \in A$ and $s_1 \in S$ such that $s \stackrel{b}{\rightarrow} s_1$, $\rho(a, b) < \theta$ and $(s_1, t_1) \in R$ (*the back condition*).

As usual, we say that s and t are λ-bisimilar, in symbols $s \sim_\lambda t$, if $(s, t) \in R$ for some λ-bisimulation R. In other words, λ-bisimilarity $\sim_\lambda$ is defined as $\sim_\lambda =_{def} \bigcup\{R : R \text{ is a } \lambda\text{-bisimulation}\}$.

Clearly, λ-bisimilarity does not always force matched actions to be identical and admits some difference between them. Such differences are captured by metrics on actions. Moreover, it is easy to see that the usual notion of bisimilarity may be regarded as λ-bisimilarity with the discrete metric (i.e., $\rho(a, b) = \infty$ for all $a \neq b$ and $\rho(a, a) = 0$ for each $a \in A$). By the way, in the case where A is finite, according to the following observation, λ-bisimilarity can be defined without referring to the variable θ.

Observation 1. Let $\sigma = (S, A, \{\stackrel{a}{\rightarrow}: a \in A\})$ be a LTS, ρ a metric on A and $\lambda \in [0, \infty)$. If A is finite then $R \subseteq S \times S$ is a λ-bisimulation if and only if for each $(s, t) \in R$ and for each $a \in A$,

(1) whenever $s \stackrel{a}{\rightarrow} s_1$ then there exist $b \in A$ and $t_1 \in S$ such that $t \stackrel{b}{\rightarrow} t_1$, $\rho(a, b) \leqslant \lambda$ and $(s_1, t_1) \in R$;

(2) whenever $t \stackrel{a}{\rightarrow} t_1$ then there exist $b \in A$ and $s_1 \in S$ such that $s \stackrel{b}{\rightarrow} s_1$, $\rho(a, b) \leqslant \lambda$ and $(s_1, t_1) \in R$.

Proof : (From right to left) Follows immediately from Definition 2.

(From left to right) To prove (1), let $R \subseteq S \times S$ be a λ−bisimulation, $(s, t) \in R$ and $s \stackrel{a}{\rightarrow} s_1$. We want to show that there exist $b \in A$ and $t_1 \in S$ such that

$$t \xrightarrow{b} t_1,\ \rho(a,\ b) \leqslant \lambda \text{ and } (s_1,\ t_1) \in R. \qquad (*)$$

To this end, we set

$$\Omega = \{\rho(a,\ c)\colon\ c \in A \text{ and } \rho(a,\ c) > \lambda\} \text{ and } \theta = \inf \Omega.$$

Firstly, we show $\theta > \lambda$. If $\Omega = \emptyset$ then $\theta = \infty > \lambda$. Otherwise, since A is finite, so is Ω. Hence, $\inf \Omega = \min \Omega > \lambda$ and $\theta > \lambda$, as desired.

Secondly, we show that there exist $b \in A$ and $t_1 \in S$ satisfying the condition $(*)$. Since R is a $\lambda-$bisimulation and $(s,\ t) \in R$ with $s \xrightarrow{a} s_1$, by Definition 2, there exist $b \in A$ and $t_1 \in S$ such that

$$t \xrightarrow{b} t_1,\ \rho(a, b) < \theta \text{ and } (s_1,\ t_1) \in R.$$

In the following, we will show $\rho(a,\ b) \leqslant \lambda$. Suppose not, then we have

$$\rho(a,\ b) \geqslant \inf \Omega = \theta.$$

This contradicts $\rho(a,\ b) < \theta$. Hence, we obtain $\rho(a,\ b) \leqslant \lambda$. Similarly, we may prove (2). □

Following Milner[7], the stratification of $\lambda-$bisimilarity is defined below.

Definition 3[11]. Let $\sigma = (S,\ A,\ \{\xrightarrow{a}\colon a \in A\})$ be a LTS and $\lambda \in [0,\ \infty)$. The function $\sharp_\sigma^\lambda$ ($\sharp^\lambda$, for short): $2^{S\times S} \rightarrow 2^{S\times S}$ is defined as follows: for all $R \subseteq S \times S$, $(s,\ t) \in \sharp^\lambda(R)$ if and only if for each $\theta > \lambda$ and for all $a \in A$, we have

(1) whenever $s \xrightarrow{a} s_1$ then there exist $b \in A$ and $t_1 \in S$ such that $t \xrightarrow{b} t_1$, $\rho(a,\ b) < \theta$ and $(s_1,\ t_1) \in R$;

(2) whenever $t \xrightarrow{a} t_1$ then there exist $b \in A$ and $s_1 \in S$ such that $s \xrightarrow{b} s_1$, $\rho(a,\ b) < \theta$ and $(s_1,\ t_1) \in R$.

Definition 4[11]. Let $\lambda \in [0,\ \infty)$. For any $\alpha \in On$, $\sim_\lambda^\alpha$ is defined inductively as follows, where On is the class of all ordinal numbers.

(1) $\sim_\lambda^0 = S \times S$;

(2) $\sim_\lambda^{\alpha+1} = \sharp^\lambda(\sim_\lambda^\alpha)$; and

(3) $\sim_\lambda^\alpha = \bigcap\limits_{\eta<\alpha} \sim_\lambda^\eta$ for any $\alpha \in On_{II}$, where On_{II} is the class of limit ordinal numbers.

Proposition 2[11]. (1) $\eta \leqslant \alpha$ implies $\sim_\lambda^\alpha \subseteq \sim_\lambda^\eta$.

(2) $\sim_\lambda = \bigcap\limits_{\alpha \in On} \sim_\lambda^\alpha$.

2.2 Logical Characterization of λ−Bisimilarity

Ying has established a logical characterization of $\lambda-$bisimilarity with the assumption that ρ is an ultra−metric or $\lambda = 0$[11,12]. In the following, we will recall this result.

Definition 5[11]. Let A be a set of labels and $\lambda \in [0, \infty)$. The modal language $ML_\lambda(A)$ is built up in terms of A and λ, and the formulae of $ML_\lambda(A)$ are recursively defined as:

$$\varphi ::= \neg\varphi \mid \bigwedge_{i\in I} \varphi_i \mid \langle a, \theta\rangle\, \varphi,$$

where $a \in A$, $\theta > \lambda$ and I is an indexing set. As usual, if the indexing set I is empty, we denote $\bigwedge_{i \in I} \varphi_i$ by tt. Moreover, we often abbreviate $ML_\lambda(A)$ to ML_λ when A is clear from the context.

Notice that infinite disjunction is admitted in the above definition. Therefore, ML_λ is an infinitary modal language.

Definition 6[11]. Let $\sigma = (S,\ A,\ \{\xrightarrow{a}: a \in A\})$ be a LTS, ρ a metric on A, and $\lambda \in [0, \infty)$. The satisfaction relation $\models$ between S and ML_λ with respect to ρ is defined inductively as follows:

(1) $s \models \langle a,\ \theta \rangle \varphi$ if there exist $b \in A$ and $s_1 \in S$ such that $s \xrightarrow{b} s_1$, $\rho(a, b) < \theta$ and $s_1 \models \varphi$;

(2) $s \models \neg\varphi$ if $s \models \varphi$ does not hold; and

(3) $s \models \bigwedge_{i \in I} \varphi_i$ if $s \models \varphi_i$ for every $i \in I$.

For convenience, we recall the notion below.

Definition 7. Let ML be a modal language. Then for all states s and t, s and t are called $ML-$equivalent, in symbols $s \equiv_{ML} t$, if they satisfy the same formulae of ML, that is, for any $\varphi \in ML$, $s \models \varphi$ iff $t \models \varphi$.

The following theorem illustrates that $\lambda-$bisimilarity coincides with $ML_\lambda-$ equivalence when ρ is an ultra$-$metric or $\lambda = 0$.

Theorem 3[11]**.** Let $\sigma = (S,\ A,\ \{\xrightarrow{a}: a \in A\})$ be a LTS, ρ a metric on A, and $\lambda \in [0,\ \infty)$. If ρ is an ultra$-$metric or $\lambda = 0$ then for each $s, t \in S$, $s \sim_\lambda t$ iff $s \equiv_{ML_\lambda} t$.

3 Logical Characterization: General Case

Theorem 3 has provided a logical characterization of $\lambda-$bisimilarity with the assumption that ρ is an ultra$-$metric or $\lambda = 0$. However, in the general metric case, this theorem does not always hold. Since $ML_\lambda-$equivalence is always an equivalence relation but, as shown by the following example, the relation $\sim_\lambda$ is not always an equivalence relation, the general logical characterization of $\sim_\lambda$ will take the form which is different from Theorem 3. This section will provide a logical characterization of $\lambda-$bisimilarity associated with a general metric.

Example 1. Let A be a set of labels and ρ a metric over A. Suppose that ρ is not an ultra$-$metric. Then there exist $a, b, c \in A$ such that $\rho(a, c) > \max\{\rho(a, b), \rho(b, c)\}$. We consider the LTS $\sigma = (\{s,\ t,\ u,\ v\},\ A,\ \{s \xrightarrow{a} v,\ t \xrightarrow{b} v,\ u \xrightarrow{c} v\})$.

Let $\lambda = \max\{\rho(a, b),\ \rho(b, c)\}$. We will show that $\sim_\lambda$ is not transitive. It is enough to prove $s \sim_\lambda t$, $t \sim_\lambda u$ and $s \not\sim_\lambda u$.

We put $R = \{(s, t),\ (t, u),\ (v, v)\}$. It is easy to check that R is a λ-bisimulation. So, we obtain $s \sim_\lambda t$ and $t \sim_\lambda u$. Next, we will show $s \not\sim_\lambda u$. To this end, we set

$$\theta = (\rho(a, c) + \lambda)/2.$$

Thus, it follows from $\rho(a,\ c) > \max\{\rho(a,\ b),\ \rho(b,\ c)\} = \lambda$ that $\rho(a, c) > \theta > \lambda$. So, since $u \xrightarrow{c} v$ and u can not reach any other state, there do not exist $d \in A$ and $u' \in S$ such that

$$\rho(a,\ d) < \theta \text{ and } u \xrightarrow{d} u'.$$

Further, it follows from $s \xrightarrow{a} v$ and Definition 2 that $s \not\sim_\lambda u$. Hence, $\sim_\lambda$ is not transitive and therefore $\sim_\lambda$ is not an equivalence relation[1] . □

In order to establish a general logical characterization of $\lambda-$bisimilarity, we introduce the modal language below, which is obtained by adding the diamond operator $\langle a \rangle$ to Ying's logical language .

Definition 8. Let A be a set of labels and $\lambda \in [0, \infty)$. The modal language $ML_\lambda^{<>}(A)$ is built up in terms of A and λ, and the formulae of $ML_\lambda^{<>}(A)$ are recursively defined as:

$$\varphi ::= \neg\varphi \mid \bigwedge_{i \in I} \varphi_i \mid \langle a \rangle \varphi \mid \langle a, \theta \rangle \varphi,$$

where $a \in A$, $\theta > \lambda$ and I is an indexing set. Similarly, we abbreviate $ML_\lambda^{<>}(A)$ to $ML_\lambda^{<>}$. The satisfaction relation for $ML_\lambda^{<>}$ is provided by adding the clause below to Definition 6.

(4) $s \models \langle a \rangle \varphi$ if there exists $s_1 \in S$ such that $s \xrightarrow{a} s_1$ and $s_1 \models \varphi$.

As usual, we define the modal nesting depth of formulae below.

Definition 9. For any $\varphi \in ML_\lambda^{<>}$, the *modal nesting depth* $d(\varphi) \in On$ of φ is defined as

(1) $d(\langle a, \theta \rangle \varphi) = d(\langle a \rangle \varphi) = d(\varphi) + 1$;

(2) $d(\neg\varphi) = d(\varphi)$; and

(3) $d(\bigwedge_{i \in I} \varphi_i) = \sup\{d(\varphi_i) : i \in I\}$.

To establish the modal characterization of $\lambda-$bisimilarity, we will introduce a binary relation over the formulae of $ML_\lambda^{<>}$, which will play a central role in the rest of this paper. Before providing its definition formally, we explain the motivation behind this notion. Recall that two states are $\lambda-$bisimilar if and only if they satisfy the forth and back conditions in Definition 2. So, in order to establish the modal characterization of $\lambda-$bisimilarity, we need to formalize these conditions in terms of the formulae of $ML_\lambda^{<>}$. According to the semantics of $ML_\lambda^{<>}$, we have the following observation.

For any state s and t, $\theta > \lambda$ and $a \in A$, $s \sim_\lambda t$ implies

(1) $s \models \langle a \rangle tt$ implies $t \models \langle a, \theta \rangle tt$ (by the forth condition)

(2) $s \models \neg \langle a, \theta \rangle tt$ implies $t \models \neg \langle a \rangle tt$ (by the back condition).

This simple observation gives us a hint about the logical characterization of $\lambda-$bisimilarity. That is, we may characterize $\lambda-$bisimilarity in terms of an appropriate binary relation H_λ over the formulae of $ML_\lambda^{<>}$, and this characterization will possess the form "$s \sim_\lambda t$ iff for any pair $(\varphi, \gamma) \in H_\lambda$, $s \models \varphi$ implies $t \models \gamma$, and vice versa". Clearly, by the above observation, H_λ should contain the pairs with the form $(\langle a \rangle tt, \langle a, \theta \rangle tt)$ and $(\neg \langle a, \theta \rangle tt, \neg \langle a \rangle tt)$. Moreover, in order to reflect the fact that the successors of $\lambda-$bisimilar states must match up to $\lambda-$bisimilarity, the formulae should admit necessary nested modal operators, that is, $(\varphi, \gamma) \in H_\lambda$ implies $(\langle a \rangle \varphi, \langle a, \theta \rangle \gamma) \in H_\lambda$. Now, we define the relation H_λ as follows.

Definition 10. The set $H_\lambda \subseteq ML_\lambda^{<>} \times ML_\lambda^{<>}$ is the smallest set of ordered pairs of formulae satisfying the following conditions:

(1) if $(\varphi, \gamma) \in H_\lambda$ then $(\neg\gamma, \neg\varphi) \in H_\lambda$;

[1] A similar observation has been obtained by Giacalone et al. in the framework of probabilistic systems[5]

(2) if $(\varphi, \gamma) \in H_\lambda$ then for each $a \in A$ and $\theta > \lambda$, $(\langle a\rangle \varphi, \langle a, \theta\rangle \gamma) \in H_\lambda$;

(3) if I is an indexing set and $(\varphi_i, \gamma_i) \in H_\lambda$ for any $i \in I$ then $(\bigwedge_{i\in I} \varphi_i, \bigwedge_{i\in I} \gamma_i) \in H_\lambda$.

Remark 1. Since $tt = \bigwedge_{i\in\emptyset} \varphi_i$, by (3), H_λ contains the pair (tt, tt). So, $H_\lambda \neq \emptyset$.

For any $s \in S$ and $(\varphi, \gamma) \in H_\lambda$, it is easy to show that $s \models \varphi$ implies $s \models \gamma$. However, the converse does not always hold. In other words, the relation H_λ is not symmetric up to logical equivalence. Moreover, notice that $(\varphi, \gamma) \in H_\lambda$ does not imply that γ is obtained from φ by replacing $\langle a\rangle$ with $\langle a, \theta\rangle$. For instance, consider $(\neg \langle a, \theta\rangle tt, \neg \langle a\rangle tt) \in H_\lambda$.

In order to express the modal characterization of the stratification of $\lambda-$ bisimilarity, we introduce the following notion.

Definition 11. For any $\alpha \in On$, H_λ^α is defined as

$$H_\lambda^\alpha =_{def} \{(\varphi, \gamma)\colon (\varphi, \gamma) \in H_\lambda, d(\varphi) \leqslant \alpha \text{ and } d(\gamma) \leqslant \alpha\}\ ^2.$$

Remark 2. It is easy to see that $H_\lambda = \bigcup_{\alpha\in On} H_\lambda^\alpha$. However, it is false that

$$H_\lambda^\alpha = \bigcup_{\eta<\alpha} H_\lambda^\eta \text{ for } \alpha \in On_{II}.$$

For instance, let $\varphi_0 = \gamma_0 = tt$, $\varphi_{i+1} = \langle a\rangle \varphi_i$ and $\gamma_{i+1} = \langle a, \theta\rangle \gamma_i$ for each $i < \omega_0$, then $(\bigwedge_{i<\omega_0} \varphi_i, \bigwedge_{i<\omega_0} \gamma_i) \in H_\lambda^{\omega_0}$ but $(\bigwedge_{i<\omega_0} \varphi_i, \bigwedge_{i<\omega_0} \gamma_i) \notin H_\lambda^i$ for each $i < \omega_0$.

In the rest of this section, we will establish the logical characterization of $\lambda-$ bisimilarity in terms of H_λ. Our proof will use the notion below, which provides a measure of the complexity of formulae.

Definition 12. For each formula φ, the *rank* $\xi(\varphi) \in On$ of φ is defined as

(1) $\xi(\neg\varphi) = \xi(\varphi) + 1$;

(2) $\xi(\langle a, \theta\rangle \varphi) = \xi(\langle a\rangle \varphi) = \xi(\varphi) + 1$; and

(3) $\xi(\bigwedge_{i\in I} \varphi_i) = \sup\{\xi(\varphi_i)\colon i \in I\} + 1$.

For each ordered pair $(\varphi, \gamma) \in H_\lambda$, we define its rank as $\xi(\varphi, \gamma) = \max\{\xi(\varphi), \xi(\gamma)\}$.

In particular, if $I = \emptyset$ then $\sup\{\xi(\varphi_i)\colon i \in I\} = \sup\emptyset = 0$. Therefore, we have $\xi(tt) = 1$. Moreover, it is easy to show that $\xi(\varphi) = \xi(\gamma)$ for each ordered pair $(\varphi, \gamma) \in H_\lambda$. So, $\xi(\varphi, \gamma) = \xi(\varphi) = \xi(\gamma)$ for each $(\varphi, \gamma) \in H_\lambda$. The following lemma is trivial but useful.

Lemma 4. For any $\varphi, \gamma \in ML_\lambda^{<>}$, if $(\varphi, \gamma) \in H_\lambda$ then

(1) φ is in one of the following forms: tt, $\langle a\rangle \psi$, $\neg\phi$ and $\bigwedge_{i\in I} \varphi_i$;

(2) if $\varphi = tt$ then $\gamma = tt$ and $\xi(\varphi, \gamma) = 1$;

(3) if $\varphi = \langle a\rangle \psi$ then there exists $\phi \in ML_\lambda^{<>}$ such that $\gamma = \langle a,\theta\rangle \phi$, $(\psi, \phi) \in H_\lambda$ and $\xi(\varphi, \gamma) = \xi(\psi, \phi) + 1$;

(4) if $\varphi = \neg\phi$ then there exists $\psi \in ML_\lambda^{<>}$ such that $\gamma = \neg\psi$, $(\psi, \phi) \in H_\lambda$ and $\xi(\varphi, \gamma) = \xi(\psi, \phi) + 1$;

(5) if $\varphi = \bigwedge_{i\in I} \varphi_i$ then there exist $\gamma_i \in ML_\lambda^{<>}$ $(i \in I)$ such that $\gamma = \bigwedge_{i\in I} \gamma_i$, $(\varphi_i, \gamma_i) \in H_\lambda$ for each $i \in I$ and $\xi(\varphi, \gamma) = \sup_{i\in I} \xi(\varphi_i, \gamma_i) + 1$.

Proof: Straightforward. □

[2]In fact, it is easy to see that $d(\varphi) = d(\gamma)$ for any $(\varphi, \gamma) \in H_\lambda$.

The proof of the lemma below will proceed by induction on the stratification of λ−bisimilarity and the rank of ordered pairs in H_λ. In order to avoid confusion, we use the notations IH_1 and IH_2 to denote two induction hypotheses, respectively.

Lemma 5. For any $\alpha \in On$ and $s, t \in S$, if $s \sim_\lambda^\alpha t$ then

(1_α) $\forall(\varphi, \gamma) \in H_\lambda^\alpha$ $(s \models \varphi \Rightarrow t \models \gamma)$;

(2_α) $\forall(\varphi, \gamma) \in H_\lambda^\alpha$ $(t \models \varphi \Rightarrow s \models \gamma)$.

Proof : We proceed by transfinite induction on α.

If $\alpha = 0$ then $H_\lambda^0 = \{(tt, tt), (\neg tt, \neg tt)\}$ (up to logical equivalence). So, the conclusion holds trivially.

Let α be an ordinal number ($\neq 0$). Suppose that

if $s \sim_\lambda^\eta t$ then (1_η) and (2_η) hold for any $s, t \in S$ and any $\eta < \alpha$. (IH_1)

In the following, we will show that the conclusion holds for α, that is, for any s, $t \in S$ such that $s \sim_\lambda^\alpha t$ and for any $(\varphi, \gamma) \in H_\lambda^\alpha$, we have

(i) $s \models \varphi$ implies $t \models \gamma$;

(ii) $t \models \varphi$ implies $s \models \gamma$.

Let $s \sim_\lambda^\alpha t$ and $(\varphi, \gamma) \in H_\lambda^\alpha$. We will show (i) and (ii) by induction on the rank of (φ, γ). By Lemma 4, $\xi(\varphi, \gamma)$ is a successor ordinal number. If $\xi(\varphi, \gamma) = 1$ then $\varphi = \gamma = tt$ and the conclusions (i) and (ii) hold trivially. Otherwise, let $\xi(\varphi, \gamma) = \zeta + 1$ and suppose that

(i) and (ii) hold for any $(\psi_1, \phi_1) \in H_\lambda^\alpha$ where $\xi(\psi_1, \phi_1) \leqslant \zeta$. (IH_2)

Let $s \models \varphi$. We will show $t \models \gamma$. By Lemma 4, it is enough to consider the following three cases.

Case 1. $\varphi = \langle a \rangle \psi$. By Lemma 4, for some $\phi \in ML_\lambda^{<>}$, we have $\gamma = \langle a, \theta \rangle \phi$, $(\psi, \phi) \in H_\lambda$ and $\xi(\psi, \phi) = \zeta$. We consider the following two cases:

Case 1.1. $\alpha = \eta + 1$. It follows from $s \models \varphi$ that $s \xrightarrow{a} s_1$ and $s_1 \models \psi$ for some $s_1 \in S_1$. Since $s \sim_\lambda^\alpha t$, for some $b \in A$ and $t_1 \in S$, we have $t \xrightarrow{b} t_1$, $\rho(a, b) < \theta$ and $s_1 \sim_\lambda^\eta t_1$. Since $d(\psi) < d(\varphi) \leqslant \alpha = \eta + 1$ and $d(\phi) < d(\gamma) \leqslant \alpha = \eta + 1$, we obtain $(\psi, \phi) \in H_\lambda^\eta$. Thus, by IH_1, we get $t_1 \models \phi$. Further, $t \xrightarrow{b} t_1$ and $\rho(a, b) < \theta$ imply $t \models \gamma$, as desired.

Case 1.2. $\alpha \in On_{II}$. Since $(\varphi, \gamma) \in H_\lambda^\alpha$, we have $d(\langle a \rangle \psi) \leqslant \alpha$ and $d(\langle a, \theta \rangle \phi) \leqslant \alpha$. Moreover, it follows from Definition 9 that $d(\langle a \rangle \psi)$ and $d(\langle a, \theta \rangle \phi)$ are successors of ordinal numbers $d(\psi)$ and $d(\phi)$ respectively. Since $\alpha \in On_{II}$, we obtain $d(\langle a \rangle \psi) < \alpha$ and $d(\langle a, \theta \rangle \phi) < \alpha$. Set

$$\mu = \max\{d(\langle a \rangle \psi), d(\langle a, \theta \rangle \phi)\}.$$

Clearly, $\mu < \alpha$ and $(\langle a \rangle \psi, \langle a, \theta \rangle \phi) \in H_\lambda^\mu$. On the other hand, since $\alpha \in On_{II}$, $s \sim_\lambda^\alpha t$ and $\mu < \alpha$, by Definition 4, we get $s \sim_\lambda^\mu t$. So, by IH_1, we have $t \models \gamma$.

Case 2. $\varphi = \neg\phi$. So, $s \not\models \phi$ implied by $s \models \varphi$. By Lemma 4, there exists $\psi \in ML_\lambda^{<>}$ such that $\gamma = \neg\psi$, $(\psi, \phi) \in H_\lambda$ and $\xi(\psi, \phi) = \zeta$. Since $d(\phi) = d(\varphi)$, $d(\psi) = d(\gamma)$ and $(\varphi, \gamma) \in H_\lambda^\alpha$, we obtain $(\psi, \phi) \in H_\lambda^\alpha$. Clearly, $t \not\models \psi$ follows from IH_2 and $s \not\models \phi$. Hence $t \models \gamma$.

Case 3. $\varphi = \bigwedge_{i \in I} \varphi_i$. By Lemma 4, there exist $\gamma_i \in ML_\lambda^{<>}$ $(i \in I)$ such that $\gamma = \bigwedge_{i \in I} \gamma_i$, $(\varphi_i, \gamma_i) \in H_\lambda$ for each $i \in I$ and $\xi(\varphi, \gamma) = \sup_{i \in I} \xi(\varphi_i, \gamma_i) + 1$. Since $\xi(\varphi, \gamma) = \zeta + 1$, $\xi(\varphi_i, \gamma_i) \leqslant \zeta$ for each $i \in I$. Moreover, for each $i \in I$, $d(\varphi_i) \leqslant d(\varphi)$ and

$d(\gamma_i) \leqslant d(\gamma)$, so, $(\varphi_i, \gamma_i) \in H_\lambda^\alpha$. Hence, by IH_2, for any $i \in I$, $t \models \gamma_i$ because of $s \models \varphi_i$. Thus we get $t \models \bigwedge_{i\in I} \gamma_i$, as desired.

Similarly, we may show that $t \models \varphi$ implies $s \models \gamma$.

Therefore, we have proved that both (i) and (ii) hold for any $(\varphi, \gamma) \in H_\lambda^\alpha$ and any $s, t \in S$ such that $s \sim_\lambda^\alpha t$. Consequently, the proof is complete. □

Lemma 6. For any $\alpha \in On$ and $s, t \in S$, if s and t satisfy the following conditions

$$(1_\alpha)\ \forall(\varphi, \gamma) \in H_\lambda^\alpha\ (s \models \varphi \Rightarrow t \models \gamma);$$
$$(2_\alpha)\ \forall(\varphi, \gamma) \in H_\lambda^\alpha\ (t \models \varphi \Rightarrow s \models \gamma),$$

then $s \sim_\lambda^\alpha t$.

Proof: We proceed by induction on α. If $\alpha = 0$ then $s \sim_\lambda^0 t$ holds trivially. If α is a limit ordinal number, then, by the induction hypothesis and $\bigcup_{\eta<\alpha} H_\lambda^\eta \subseteq H_\lambda^\alpha$, we get $s \sim_\lambda^\eta t$ for each $\eta < \alpha$. So, $s \sim_\lambda^\alpha t$ follows from $\sim_\lambda^\alpha = \bigcap_{\eta<\alpha} \sim_\lambda^\eta$.

Now we consider the nontrivial case where α is a successor ordinal number. Suppose that $\alpha = \eta + 1$ and the conclusion holds for η. Let s and t satisfy conditions (1_α) and (2_α). We will show that $s \sim_\lambda^\alpha t$. Otherwise, without loss of generality we may assume that, for some $\theta > \lambda$, $a \in A$ and $s_1 \in S$ such that $s \xrightarrow{a} s_1$, we have

$$\text{for each } b \in A,\ t_1 \in S, \text{ if } t \xrightarrow{b} t_1 \text{ and } \rho(a, b) < \theta \text{ then } s_1 \not\sim_\lambda^\eta t_1.$$

We set

$$\Delta = \{t_1 \in S : b \in A,\ \rho(a, b) < \theta \text{ and } t \xrightarrow{b} t_1\}.$$

We demonstrate the following claim first.

Claim. For each $t_1 \in \Delta$, there exists $(\varphi, \gamma) \in H_\lambda^\eta$ such that $s_1 \models \varphi$ and $t_1 \models \neg\gamma$.

Let $t_1 \in \Delta$. Since $s_1 \not\sim_\lambda^\eta t_1$, by the induction hypothesis, there exists $(\psi, \phi) \in H_\lambda^\eta$ satisfying at least one of the following conditions:

(i) $s_1 \models \psi$ and $t_1 \models \neg\phi$.

(ii) $t_1 \models \psi$ and $s_1 \models \neg\phi$.

The claim holds trivially if condition (i) holds. If condition (ii) holds, we have $t_1 \models \neg\neg\psi$ and $s_1 \models \neg\phi$. Clearly, $(\neg\phi, \neg\psi) \in H_\lambda$ follows from $(\psi, \phi) \in H_\lambda$. Moreover, since $d(\neg\psi) = d(\psi) \leqslant \eta$ and $d(\neg\phi) = d(\phi) \leqslant \eta$, we get $(\neg\phi, \neg\psi) \in H_\lambda^\eta$, as desired.

Now, we return to the proof of this lemma. By the above claim, for every $v \in \Delta$, we may choose one pair $(\varphi_v, \gamma_v) \in H_\lambda^\eta$ such that $s_1 \models \varphi_v$ and $v \models \neg\gamma_v$. Clearly, $(\bigwedge_{v\in\Delta} \varphi_v, \bigwedge_{v\in\Delta} \gamma_v) \in H_\lambda$. Moreover, since $d(\varphi_v) \leqslant \eta$ and $d(\gamma_v) \leqslant \eta$ for each $v \in \Delta$, we get

$$d(\bigwedge_{v\in\Delta} \varphi_v) = \sup_{v\in\Delta} d(\varphi_v) \leqslant \eta \text{ and } d(\bigwedge_{v\in\Delta} \gamma_v) = \sup_{v\in\Delta} d(\gamma_v) \leqslant \eta.$$

Hence,

$$(\bigwedge_{v\in\Delta} \varphi_v, \bigwedge_{v\in\Delta} \gamma_v) \in H_\lambda^\eta \text{ and } (\langle a\rangle \bigwedge_{v\in\Delta} \varphi_v, \langle a,\theta\rangle \bigwedge_{v\in\Delta} \gamma_v) \in H_\lambda^\alpha.$$

Since $s \xrightarrow{a} s_1$ and $s_1 \models \varphi_v$ for each $v \in \Delta$, we get

$$s \models \langle a \rangle \bigwedge_{v \in \Delta} \varphi_v.$$

Consequently,

$$t \models \langle a, \theta \rangle \bigwedge_{v \in \Delta} \gamma_v.$$

Thus, for some $t_1 \in \Delta$, we have $t_1 \models \bigwedge_{v \in \Delta} \gamma_v$. So, $t_1 \models \gamma_{t_1}$, a contradiction. □

Then we obtain the main result of this section as follows.

Theorem 7. Let $\sigma = (S, A, \{\xrightarrow{a}: a \in A\})$ be a LTS and ρ a metric on A. For any $\alpha \in On$, $s, t \in S$ and $\lambda \in [0, \infty)$, $s \sim_\lambda^\alpha t$ if and only if the following conditions hold

(1_α) $\forall(\varphi, \gamma) \in H_\lambda^\alpha$ $(s \models \varphi \Rightarrow t \models \gamma)$;

(2_α) $\forall(\varphi, \gamma) \in H_\lambda^\alpha$ $(t \models \varphi \Rightarrow s \models \gamma)$.

Proof: Follows from Lemma 5 and 6. □

As an immediate consequence of Theorem 7, we get

Corollary 8. Let $\sigma = (S, A, \{\xrightarrow{a}: a \in A\})$ be a LTS and ρ a metric on A. For any $s, t \in S$ and $\lambda \in [0, \infty)$, $s \sim_\lambda t$ if and only if the following conditions hold

(1) $\forall(\varphi, \gamma) \in H_\lambda$ $(s \models \varphi \Rightarrow t \models \gamma)$, and

(2) $\forall(\varphi, \gamma) \in H_\lambda$ $(t \models \varphi \Rightarrow s \models \gamma)$.

Proof: Follows immediately from Proposition 2 and Theorem 7. □

4 A Note on Ultra-Metric Case

In the previous section, a logical characterization of $\lambda-$bisimilarity has been obtained. Because λ-bisimilarity is not always an equivalence relation, the form of this logical characterization is different from usual ones in the literature[6,11,12]. However, in the ultra$-$metric case, Ying has shown that $\lambda-$bisimilarity is an equivalence relation[11]. A natural question raised at this point is: whether $\lambda-$bisimilarity coincides with $ML_\lambda^{<>}-$equivalence when the metric is an ultra$-$metric. The following simple example gives a negative answer.

Example 2. Let $A = \{a, b\}$ and $\lambda \in (0, \infty)$. We consider the LTS $\sigma = (\{s, t, v\}, A, \{s \xrightarrow{a} v, t \xrightarrow{b} v\})$ and define the distance function ρ on A as: for any $x, y \in A$, if $x = y$ then $\rho(x, y) = 0$ else $\rho(x, y) = \lambda$. It is easy to show that ρ is an ultra$-$metric. In the following, we will show $s \sim_\lambda t$ but $s \not\equiv_{ML_\lambda^{<>}} t$.

We put $R = \{(s, t), (v, v)\}$. It is easy to check that R is a $\lambda-$bisimulation. Thus, we get $s \sim_\lambda t$. Next, we show $s \not\equiv_{ML_\lambda^{<>}} t$. It follows from $s \xrightarrow{a} v$ that

$$s \models \langle a \rangle\, tt.$$

On the other hand, since $t \xrightarrow{b} v$ is the unique transition of t and $a \neq b$, we obtain

$$t \not\models \langle a \rangle\, tt.$$

So, s and t are not $ML_\lambda^{<>}-$equivalent, although they are $\lambda-$bisimilar. □

In fact, according to Hennessy-Milner theorem[6], it is easy to show that $ML_\lambda^{<>}-$equivalence coincides with bisimilarity, rather than $\lambda-$bisimilarity. So, the logical characterization obtained in the previous section can not degenerate into one in the usual style even if the metric is an ultra-metric.

In order to overcome this defect, this section will provide another logical characterization of $\lambda-$bisimilarity and will demonstrate that, in the ultra$-$metric case, this characterization degenerates into one in the usual style. Moreover, it coincides with one obtained by Ying in[11, 12]. To this end, the following auxiliary definition is needed.

Definition 13. Let ρ be a metric on A and a, $b \in A$. For any $\theta > 0$, a and b are said to be $\theta - similar$, in symbols $a \simeq_{\rho,\theta} b$, if they have the same $\theta-$ neighborhood with respect to ρ, that is, they satisfy the condition below

$$\forall c \in A(\rho(a, c) < \theta \Leftrightarrow \rho(b, c) < \theta).$$

In the following, we abbreviate $a \simeq_{\rho,\ \theta} b$ to $a \simeq_\theta b$ if we know ρ from the context.

Observation 9. Let ρ be a metric on A and a, $b \in A$. Then for any $\theta > 0$, $a \simeq_\theta b$ implies $\rho(a,\ b) < \theta$. In particular, if ρ is an ultra$-$metric then the converse also holds.

Proof: Straightforward. □

The following simple observation provides another characterization of $\lambda-$ bisimilarity, which throws a light on pursuing a new logical characterization of $\lambda-$bisimilarity.

Observation 10. Let $\sigma = (S,\ A,\ \{\stackrel{a}{\to}: a \in A\})$ be a LTS, ρ a metric on A, and $\lambda \in [0, \infty)$. Then, for any s, $t \in S$, $s \sim_\lambda t$ if and only if for each $a \in A$ and for each $\theta > \lambda$,

(1) whenever $s \stackrel{b}{\to} s_1$ for some $b \in A$ such that $a \simeq_{\ \theta} b$, then there exist $c \in A$ and $t_1 \in S$ such that $t \stackrel{c}{\to} t_1$, $\rho(a,\ c) < \theta$ and $s_1 \sim_\lambda t_1$;

(2) whenever $t \stackrel{b}{\to} t_1$ for some $b \in A$ such that $a \simeq_{\ \theta} b$, then there exist $c \in A$ and $s_1 \in S$ such that $s \stackrel{c}{\to} s_1$, $\rho(a,\ c) < \theta$ and $s_1 \sim_\lambda t_1$.

Proof: Straightforward. □

In order to obtain the logical characterization of $\lambda-$bisimilarity, we want to formalize conditions (1) and (2) in the above observation in terms of logical formulae. However, it seems impossible to formalize these conditions only using the modal operator $\langle a, \theta\rangle$. So, we add the modal operator $\ll a,\ \theta \gg$ to Ying's logical language. Formally, the modal language considered in this section is defined as follows.

Definition 14. Let A be a set of labels and $\lambda \in [0,\ \infty)$. The modal language $ML_\lambda^{\ll\gg}(A)$ is built up in terms of A and λ, and the formulae of $ML_\lambda^{\ll\gg}(A)$ are recursively defined as:

$$\varphi ::= \neg\varphi \mid \bigwedge_{i\in I} \varphi_i \mid \ll a,\ \theta \gg \varphi \mid \langle a, \theta\rangle\, \varphi,$$

where $a \in A$, $\theta > \lambda$ and I is an indexing set. Similarly, we abbreviate $ML_\lambda^{\ll\gg}(A)$ to $ML_\lambda^{\ll\gg}$. The satisfaction relation for $ML_\lambda^{\ll\gg}$ is provided by adding the clause below to Definition 6.

(4) $s \models \ll a,\ \theta \gg \varphi$ if there exist $b \in A$ and $s_1 \in S$ such that $s \stackrel{b}{\to} s_1$, $a \simeq_{\ \theta} b$ and $s_1 \models \varphi$.

Similar to H_λ, the binary relation $E_\lambda \subseteq ML_\lambda^{\ll\gg} \times ML_\lambda^{\ll\gg}$ is defined as

Definition 15. The set $E_\lambda \subseteq ML_\lambda^{\ll\gg} \times ML_\lambda^{\ll\gg}$ is the smallest set of ordered pairs of formulae satisfying the following conditions:

(1) if $(\varphi,\ \gamma) \in E_\lambda$ then $(\neg\gamma,\ \neg\varphi) \in E_\lambda$;

(2) if $(\varphi, \gamma) \in E_\lambda$ then for each $a \in A$ and $\theta > \lambda$, $(\ll a, \theta \gg \varphi, \langle a, \theta \rangle \gamma) \in E_\lambda$;

(3) if I is an indexing set and $(\varphi_i, \gamma_i) \in E_\lambda$ for any $i \in I$ then $(\bigwedge_{i\in I} \varphi_i, \bigwedge_{i\in I} \gamma_i) \in E_\lambda$.

For any $\alpha \in On$, $E_\lambda^\alpha =_{def} \{(\varphi, \gamma) \in E_\lambda$: $d(\varphi) \leqslant \alpha$ and $d(\gamma) \leqslant \alpha\}$, where $d(\varphi)$ and $d(\gamma)$ are the modal nesting depth of φ and γ respectively which are defined similarly to Definition 9.

We also can provide the modal characterization of $\lambda-$bisimilarity in terms of E_λ. Formally, we have the following theorem.

Theorem 11. Let $\sigma = (S, A, \{\xrightarrow{a}: a \in A\})$ be a LTS and ρ a metric on A. For any $\alpha \in On$, $s, t \in S$ and $\lambda \in [0, \infty)$, $s \sim_\lambda^\alpha t$ if and only if the following conditions hold

(1_α) $\forall(\varphi, \gamma) \in E_\lambda^\alpha$ $(s \models \varphi \Rightarrow t \models \gamma)$;

(2_α) $\forall(\varphi, \gamma) \in E_\lambda^\alpha$ $(t \models \varphi \Rightarrow s \models \gamma)$.

The proof of the above theorem is similar to Theorem 7. We leave it to the interested reader. In the following, we will illustrate that for any ultra$-$metric ρ, the above result degenerates into one in the usual style. Before demonstrating it, we introduce some auxiliary notions below.

Definition 16. A function $\delta : ML_\lambda^{\ll\gg} \to ML_\lambda$ is defined inductively as

(1) $\delta(tt) = tt$

(2) $\delta(\neg\varphi) = \neg\delta(\varphi)$

(3) $\delta(\ll a, \theta \gg \varphi) = \langle a, \theta \rangle\, \delta(\varphi)$

(4) $\delta(\langle a, \theta \rangle\, \varphi) = \langle a, \theta \rangle\, \delta(\varphi)$

(5) $\delta(\bigwedge_{i\in I} \varphi_i) = \bigwedge_{i\in I} \delta(\varphi_i)$

Clearly, for each formula φ of $ML_\lambda^{\ll\gg}$, $\delta(\varphi)$ is obtained from φ by replacing all modal operators $\ll a, \theta \gg$ with the corresponding operators $\langle a, \theta \rangle$. It will be shown that $\delta(\varphi)$ and φ are logically equivalent in the case of ultra$-$metric (see Observation 12).

In the ultra$-$metric case, in order to show that Theorem 11 degenerates into one in the usual style (i.e., $\sim_\lambda = \equiv_{ML_\lambda^{\ll\gg}}$), we need to prove the following results:

(i) for any $(\varphi, \gamma) \in E_\lambda$, φ and γ are logically equivalent;

(ii) for any formula φ in $ML_\lambda^{\ll\gg}$, there exists $(\varphi_1, \varphi_2) \in E_\lambda$ such that φ_1, φ_2 and φ are logically equivalent.

To this end, we introduce the pair of mappings δ_1 and δ_2 as follows. We will prove that for each φ in $ML_\lambda^{\ll\gg}$, the pair $(\delta_1(\varphi), \delta_2(\varphi))$ satisfies the condition (ii) above.

Definition 17. Two functions $\delta_1, \delta_2 : ML_\lambda^{\ll\gg} \to ML_\lambda^{\ll\gg}$ are defined inductively as

(1) $\delta_1(tt) = \delta_2(tt) = tt$

(2) $\delta_1(\neg\varphi) = \neg\delta_2(\varphi)$ $\qquad$ $\delta_2(\neg\varphi) = \neg\delta_1(\varphi)$

(3) $\delta_1(\ll a, \theta \gg \varphi) = \delta_1(\langle a, \theta \rangle\, \varphi) = \ll a, \theta \gg \delta_1(\varphi)$

(4) $\delta_2(\ll a, \theta \gg \varphi) = \delta_2(\langle a, \theta \rangle\, \varphi) = \langle a, \theta \rangle\, \delta_2(\varphi)$

(5) $\delta_1(\bigwedge_{i\in I} \varphi_i) = \bigwedge_{i\in I} \delta_1(\varphi_i)$ $\qquad$ $\delta_2(\bigwedge_{i\in I} \varphi_i) = \bigwedge_{i\in I} \delta_2(\varphi_i)$

Observation 12. For any formula $\varphi \in ML_\lambda^{\ll\gg}$ and $s \in S$, we have

(1) $(\delta_1(\varphi), \delta_2(\varphi)) \in E_\lambda$.

(2) If ρ is an ultra$-$metric then $s \models \varphi$ iff $s \models \delta_1(\varphi)$ iff $s \models \delta_2(\varphi)$.

(3) If ρ is an ultra$-$metric then $s \models \varphi$ iff $s \models \delta(\varphi)$.

(4) $d(\varphi) = d(\delta(\varphi)) = d(\delta_1(\varphi)) = d(\delta_2(\varphi))$.

Proof: Induction on the structure of φ. For (2) and (3), when φ is of the form either $\ll a, \theta \gg \psi$ or $\langle a, \theta\rangle \psi$, appeal to Observation 9. □

Observation 13. If ρ is an ultra−metric then for any $s \in S$ and for any $(\varphi, \gamma) \in E_\lambda$, $s \models \varphi$ iff $s \models \gamma$.

Proof: Induction on the rank of (φ, γ). When φ is of the form $\ll a, \theta \gg \psi$, use Observation 9. □

The following result provides Hennessy−Milner logical characterizations of λ-bisimilarity associated with an ultra−metric. Amongst, the conclusions (3) and (4) have been obtained by Ying in[11, 12]. In particular, if $\lambda = 0$ then (3) and (4) also hold for λ−bisimilarity associated with any metric[11,12].

Corollary 14. Let ρ be an ultra−metric and $\lambda \in [0, \infty)$. Then for each s, $t \in S$, we have

(1) for any $\alpha \in On$, $s \sim_\lambda^\alpha t$ iff $s \equiv_{ML_{\lambda,\alpha}^{\ll\gg}} t$, where $ML_{\lambda,\alpha}^{\ll\gg} = \{\varphi \in ML_\lambda^{\ll\gg} : d(\varphi) \leqslant \alpha\}$,

(2) $s \sim_\lambda t$ iff $s \equiv_{ML_\lambda^{\ll\gg}} t$,

(3) for any $\alpha \in On$, $s \sim_\lambda^\alpha t$ iff $s \equiv_{ML_\lambda^\alpha} t$, where $ML_\lambda^\alpha = \{\varphi \in ML_\lambda : d(\varphi) \leqslant \alpha\}$,

(4) $s \sim_\lambda t$ iff $s \equiv_{ML_\lambda} t$.

Proof: Since (2) and (4) follow immediately from (1) and (3) respectively, it suffices to prove (1) and (3).

(1) Let $\alpha \in On$ and $s, t \in S$. First, we show $s \sim_\lambda^\alpha t$ implies $s \equiv_{ML_{\lambda,\alpha}^{\ll\gg}} t$. Suppose that $s \sim_\lambda^\alpha t$ and $\varphi \in ML_{\lambda,\alpha}^{\ll\gg}$. Then we have

$s \models \varphi$ iff $s \models \delta_1(\varphi)$ and $s \models \delta_2(\varphi)$
((2) in Observation 12)
iff $t \models \delta_1(\varphi)$ and $t \models \delta_2(\varphi)$
((1), (2) and (4) in Observation 12, Theorem 11)
iff $t \models \varphi$
((2) in Observation 12)

Hence, $s \equiv_{ML_{\lambda,\alpha}^{\ll\gg}} t$. Next, we prove the converse. Suppose that $s \equiv_{ML_{\lambda,\alpha}^{\ll\gg}} t$. We will show that s and t satisfy conditions (1_α) and (2_α) in Theorem 11. Let $\psi, \phi \in ML_\lambda^{\ll\gg}$ such that $(\psi, \phi) \in E_\lambda^\alpha$. Suppose that $s \models \psi$. Then, by $s \equiv_{ML_{\lambda,\alpha}^{\ll\gg}} t$, we obtain $t \models \psi$. So, by Observation 13, it follows from $(\psi, \phi) \in E_\lambda$ that $t \models \phi$. Similarly, we may show that $t \models \psi$ implies $s \models \phi$. Thus, by Theorem 11, we get $s \sim_\lambda^\alpha t$, as desired.

(3) Let $\alpha \in On$ and $s, t \in S$. Since $ML_\lambda \subseteq ML_\lambda^{\ll\gg}$, we obtain $ML_\lambda^\alpha \subseteq ML_{\lambda,\alpha}^{\ll\gg}$. Then, it follows from (1) that $s \sim_\lambda^\alpha t$ implies $s \equiv_{ML_\lambda^\alpha} t$. In the following, we show the converse. Suppose that $s \equiv_{ML_\lambda^\alpha} t$. We will show $s \equiv_{ML_{\lambda,\alpha}^{\ll\gg}} t$. Let $\varphi \in ML_{\lambda,\alpha}^{\ll\gg}$. Then, we have

$s \models \varphi$ iff $s \models \delta(\varphi)$
((3) in Observation 12)
iff $t \models \delta(\varphi)$
((4) in Observation 12 and $s \equiv_{ML_\lambda^\alpha} t$)
iff $t \models \varphi$
((3) in Observation 12).

So, we get $s \equiv_{ML_{\lambda,\alpha}^{\ll\gg}} t$. Thus, it follows from (1) that $s \sim_\lambda^\alpha t$. □

5 Discussion

This paper generalizes the logical characterization of $\lambda-$bisimilarity obtained by Ying. We consider two languages $ML_\lambda^{<>}$ and $ML_\lambda^{\ll\gg}$ which are obtained from Ying's logical language by adding $\langle a\rangle$ and $\ll a, \theta \gg$ respectively. Thus, it seems that there exist a number of modal languages that may be used to characterize $\lambda-$bisimilarity. A family of such languages is described below in terms of modal operators $\langle a\rangle$ and $\ll a, \theta \gg$.

Let A be a set of labels, ρ a metric on A and $\lambda \in [0, \infty)$. Suppose that $ML^*{}_\lambda$ is a modal language obtained by adding the modal operator $*(a, \theta)$ to Ying's logical language, and the semantic interpretation for $*(a, \theta)$ with respect to ρ satisfies the following conditions:

(1*) if $s \xrightarrow{a} s_1$ and $s_1 \models \varphi$ then $s \models *(a,\theta)\varphi$;

(2*) if $s \models *(a,\theta)\varphi$ then there exist $b \in A$ and $s_1 \in S$ such that $s \xrightarrow{b} s_1$, $a \simeq_\theta b$ and $s_1 \models \varphi$.

Clearly, both the modal operator $\langle a\rangle$ and $\ll a, \theta \gg$ satisfy (1*) and (2*). It is easy to see that $s \models \langle a\rangle\, \varphi$ implies $s \models *(a,\theta)\varphi$ and $s \models *(a,\theta)\varphi$ implies $s \models \ll a,\theta \gg \varphi$ for any state s and formula φ. Furthermore, we have the following observation.

Observation 17. Let ρ be a metric and $\lambda \in [0, \infty)$. Then, for any $(\varphi, \gamma) \in E_\lambda$ and $s, t \in S$, we have

(1) $s \models \varphi_{<>} \Rightarrow s \models \varphi_*$ and $s \models \varphi_* \Rightarrow s \models \varphi$,

(2) $s \models \gamma \Rightarrow s \models \gamma_*$ and $s \models \gamma_* \Rightarrow s \models \gamma_{<>}$,

(3) $(s \models \varphi \Rightarrow t \models \gamma) \Rightarrow (s \models \varphi_* \Rightarrow t \models \gamma_*)$,

(4) $(s \models \varphi_* \Rightarrow t \models \gamma_*) \Rightarrow (s \models \varphi_{<>} \Rightarrow t \models \gamma_{<>})$,

where (φ_*, γ_*) and $(\varphi_{<>}, \gamma_{<>})$ are obtained from (φ, γ) by replacing $\ll a, \theta \gg$ with $*(a, \theta)$ and $\langle a\rangle$ respectively.

Proof: Induction on the rank of (φ, γ). □

From the above observation together with Theorem 7 and 11, it is easy to derive a logical characterization of $\lambda-$bisimilarity in terms of $ML^*{}_\lambda$ satisfying (1*) and (2*). The further properties of the family of such modal languages will be explored in the further work.

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